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Replica symmetry breaking of the Ising spin-glass with finite connectivity

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Abstract. We consider the Ising spin-glass on a lattice with finite connectivity ($= M + 1$). Using the recent method of large connectivity expansion, the free energies at finite temperatures are evaluated numerically at first-step replica symmetry breaking (RSB). The $1/M$ expansion at finite temperature diverges as $T \to 0$ as in the replica symmetric case but is well behaved for a larger range. The $1/\sqrt{M}$ expansion at zero temperature is also calculated at higher steps of RSB. Expression for the free-energy expansion is derived for an arbitrary step of RSB up to order $1/M$. Explicit numerical values at second-step RSB are obtained. From our results, we speculate that the divergence in the finite-temperature expansion might disappear for the exact infinite-step RSB solution. We also compare our results with the simulation results of graph bipartitioning.

1. Introduction

Ever since Parisi proposed his replica symmetry breaking (RSB) solution [1] to the Sherrington–Kirkpatrick [2] (SK) infinite range spin-glass model, there have been many efforts [3] to extend the theory for the more realistic short-range Bravais lattices. While there is a general agreement on the Parisi RSB solution to the infinte range spin-glass, there are many controversies and questions about whether the features in the Parisi solution, like the coexistence of many thermodynamic states in the spin-glass phase, remain correct for the short-range real spin-glasses. Recently, much interest has been focused on the theory of spin-glasses on lattices with finite connectivity [4–10]. These systems are closer in nature to the real spin-glasses because of the finite valence of the Bravais lattices. Moreover, spin-glasses on such random lattices with finite connectivity are directly related to some well known optimisation problems like graph colouring [8] and partitioning [6–9].

In this paper, we consider random lattices with fixed connectivity equal to $M + 1$, i.e. each site is connected to $M + 1$ other sites. The spin-glass Hamiltonian is given by

$$\mathcal{H} = -\sum_{(ij)} J_{ij}\sigma_i\sigma_j \qquad (1)$$

where $\sigma_i = \pm 1$ is the Ising spin at the ith site and the bonds J_{ij} can be positve or negative and obey the independent distribution $\rho(J_{ij})$. Goldschmidt and DeDominicis [11, 12] recently proposed a way to construct an RSB solution and presented the first-step RSB solution for such systems. The method involved a systematic expansion in inverse powers of the connectivity. Following their method, we present new first-step RSB numerical results for the free energy at finite temperatures in section 2. In section 3, we demonstrate how to construct large connectivity expansion solutions at zero temperature for higher step RSB and give explicit results for second-step RSB.

2. Finite temperatures and the first-step RSB results

The details of the expansion method are presented in [12], we will just outline the scheme here. The useful quantity in the large connectivity expansion scheme is the global order parameter [13] $g(\{\sigma_\alpha\})$ satisfying the recursion relation

$$g_n(\{\sigma_\alpha\}) = \int \mathrm{d}J\,\rho(J)\frac{\mathrm{Tr}_\tau g_n^M(\{\tau_\alpha\})\exp(\beta J\sum_\alpha \sigma_\alpha\tau_\alpha)}{\mathrm{Tr}\, g_n^M(\{\tau_\alpha\})} \tag{2}$$

where $\alpha = 1,2\ldots n$ is the replica index, the τ are Ising spin variables and ultimately the limit $n \to 0$ is taken. The order parameters $q_{\alpha_1\cdots\alpha_r}$ is given by

$$q_{\alpha_1\ldots\alpha_r} = \mathrm{Tr}\, g^M(\{\sigma_\alpha\})\sigma_{\alpha_1}\ldots\sigma_{\alpha_r}/\mathrm{Tr}\, g^M(\{\sigma_\alpha\})\,. \tag{3}$$

The free-energy density at any temperature has been shown [12] to be

$$n\beta f = M\ln g_n^{M+1}(\{\sigma_\alpha\}) - \frac{M+1}{2} \times \ln\left\{\int \mathrm{d}J\,\rho(J)\mathrm{Tr}_\sigma\mathrm{Tr}_\tau g_n^M(\{\sigma_\alpha\})g_n^M(\{\tau_\alpha\})\exp\left(\beta J\sum_\alpha \sigma_\alpha\tau_\alpha\right)\right\}. \tag{4}$$

The expansion method works for general $\rho(J)$, but for simplicity $\rho(J)$ is chosen to be

$$\rho(J) = \frac{\delta(J-J_0)+\delta(J+J_0)}{2}\,. \tag{5}$$

g_n^M can be expanded in powers of $1/M$ for large values of M with the scaling $J_0 = \tilde{J}/\sqrt{M}$

$$g_n^M(\{\sigma_\alpha\}) = \cosh^{nM}(\lambda/\sqrt{M})\left[1+\frac{\lambda^2}{M}\left(1-\frac{2\lambda^2}{3M}\right)\sum_{(\alpha,\beta)} q_{\alpha\beta}\sigma_\alpha\sigma_\beta + \frac{\lambda^4}{M^2}\sum_{(\alpha\beta\gamma\delta)} q_{\alpha\beta\gamma\delta}\sigma_\alpha\sigma_\beta\sigma_\gamma\sigma_\delta + \cdots\right]^M \tag{6}$$

where $\lambda \equiv \beta\tilde{J}$. The order parameters $q_{\alpha_1\ldots\alpha_r}$ are also expanded in $1/M$ as

$$q_{\alpha_1\ldots\alpha_r} = q^{(0)}_{\alpha_1\ldots\alpha_r} + q^{(1)}_{\alpha_1\ldots\alpha_r}/M + \cdots\,. \tag{7}$$

Substituting these expressions into the free-energy density, the latter can be expanded as

$$\beta f = \beta f_0 + \frac{\beta f_1}{M} + \mathcal{O}\left(\frac{1}{M^2}\right) \tag{8}$$

with

$$\beta f_0 = -\frac{\lambda^2}{4} + \frac{\lambda^2}{2n}\sum_{(\alpha\beta)} q_{\alpha\beta}^{(0)2} - \frac{1}{n}\ln \mathrm{Tr}\,\exp\Big(\lambda^2 \sum_{(\alpha\beta)} q_{\alpha\beta}^{(0)}\sigma_\alpha\sigma_\beta\Big) \tag{9}$$

$$\beta f_1 = -\frac{\lambda^2}{4} + \frac{\lambda^4}{24} - \frac{\lambda^2}{2n}\Big(1-\frac{5\lambda^2}{3}\Big)\sum_{(\alpha\beta)} q_{\alpha\beta}^{(0)2} - \frac{\lambda^4}{2n}\sum_{(\alpha\beta\gamma\delta)} q_{\alpha\beta\gamma\delta}^{(0)2}$$
$$+ \frac{3\lambda^4}{n}\sum_{(\alpha\beta\gamma)} q_{\alpha\beta}^{(0)}q_{\beta\gamma}^{(0)}q_{\gamma\alpha}^{(0)} + \frac{\lambda^4}{n}\sum_{(\alpha\beta\gamma\delta)}(q_{\alpha\beta}^{(0)}q_{\gamma\delta}^{(0)} + 2\,\mathrm{perm})q_{\alpha\beta\gamma\delta}^{(0)} \tag{10}$$

where the limit $n \to 0$ is understood. Up to this order, the free energy depends only on the $q^{(0)}$, not on the $q^{(1)}$ and thus for convenience the superscript $^{(0)}$ will be dropped in what follows in this section. Also the q with higher numbers of replica indices will only appear in higher orders in $1/M$.

It has been shown that the replica symmetric (RS) solution is unstable [13] below the spin-glass transition temperature and first-step RSB has been consider ed in [11, 12]. The RSB scheme is the same as Parisi's case of the SK model. For first-step RSB, the replica index α is parametrised as $\alpha = (K,\gamma)$ where $K = 1,2,\ldots,n/m$ is the box label and $\gamma = 1,2,\ldots,m$ is the label inside a box. The values of $q_{\alpha_1\ldots\alpha_r}$ are classified according to the number of replica indices in the same box. For $q_{\alpha_1\alpha_2}$, the possible values are q_2 and q_{11}. For $q_{\alpha_1\alpha_2\alpha_3\alpha_4}$, the possible values are $q_4, q_{22}, q_{31}, q_{211}$ and q_{1111}. Higher steps of RSB are described in next section. The expressions for f_0 and f_1 for first-step RSB have been given in [12]

$$\beta f_0 = -\frac{\lambda^2}{4}(1 + mq_{11}^2 + (1-m)q_2^2 - 2q_2) - \ln 2 - \frac{1}{m}\int Dz \ln \int Dy\,\cosh^m\phi \tag{11}$$

$$\beta f_1 = -\frac{\lambda^2}{4} + \frac{\lambda^4}{24} - \frac{\lambda^2}{4}\Big(1-\frac{5\lambda^2}{3}\Big)[(m-1)q_2^2 - mq_{11}^2]$$
$$-\frac{\lambda^4}{48}[(m-1)(m-2)(m-3)q_4^2 - 3m(m-1)^2q_{22}^2 - 4m(m-1)(m-2)q_{31}^2$$
$$+12m(m-1)q_{211}^2 - 6m^3q_{1111}^2] + \frac{\lambda^2}{2}[(m-1)(m-2)q_2^3 - 3m(m-1)q_2q_{11}^2$$
$$+2m^2q_{11}^3] + \frac{\lambda^4}{8}[(m-1)(m-2)(m-3)q_4q_2^2 - m(m-1)^2q_{22}(q_2^2+2q_{11}^2)$$
$$-4m(m-1)(m-2)q_{31}q_2q_{11} + 4m^2(m-1)q_{211}(q_2q_{11}+2q_{11}^2) - 6m^3q_{1111}q_{11}^2] \tag{12}$$

where $Dz \equiv (\mathrm{d}z/\sqrt{2\pi})\exp(-z^2/2)$ and $\phi \equiv \lambda z\sqrt{q_{11}} + \lambda y\sqrt{q_2 - q_{11}}$. The q satisfy the saddle point equations (3), and for first-step RSB they read

$$q_2 = \int Dz\,\frac{\int Dy\,\cosh^m\phi\tanh^2\phi}{\int Dy\,\cosh^m\phi} \tag{13a}$$

$$q_{11} = \int Dz\left(\frac{\int Dy\,\cosh^m\phi\tanh\phi}{\int Dy\,\cosh^m\phi}\right)^2 \tag{13b}$$

$$q_4 = \int Dz \frac{\int Dy \cosh^m \phi \tanh^4 \phi}{\int Dy \cosh^m \phi} \tag{13c}$$

$$q_{22} = \int Dz \left(\frac{\int Dy \cosh^m \phi \tanh^2 \phi}{\int Dy \cosh^m \phi}\right)^2 \tag{13d}$$

$$q_{31} = \int Dz \frac{(\int Dy \cosh^m \phi \tanh^3 \phi)(\int Dy \cosh^m \phi \tanh \phi)}{(\int Dy \cosh^m \phi)^2} \tag{13e}$$

$$q_{211} = \int Dz \frac{(\int Dy \cosh^m \phi \tanh^2 \phi)(\int Dy \cosh^m \phi \tanh \phi)^2}{(\int Dy \cosh^m \phi)^3} \tag{13f}$$

$$q_{1111} = \int Dz \left(\frac{\int Dy \cosh^m \phi \tanh \phi}{\int Dy \cosh^m \phi}\right)^4 . \tag{13g}$$

After solving q_{11}, q_2 and m from the equations $\partial f_0/\partial q_{11} = \partial f_0/\partial q_2 = \partial f_0/\partial m = 0$, which are obtained from extremising f_0, $q_4, q_{22}, q_{31}, q_{211}, q_{1111}$ and thus the free-energy density can be evaluated. The integrals are numerically evaluated using Hermite quadratures up to a hundred points. The RS case can be recovered by setting $q_{11} = q_2$ and the results agree with those presented in [11] and [12]. The new results for RSB at finite temperatures are given here. Figure 1 shows the variation of the scaled free-energy density as a function of the scaled temperature with and without RSB. The rescaled free energy $f/J_0\sqrt{M}$ is well behaved up to $T/J_0\sqrt{10} \simeq 0.2$ for the RS case. It appears to diverge as the temperature is further lowered. When first-step RSB is introduced, the free energy is well behaved up to $T/J_0\sqrt{10} \simeq 0.1$ and diverges afterwards. This divergence is because of the fact that the expansion parameter should be in powers of $1/\sqrt{M}$ instead of $1/M$ at $T = 0$. Nevertheless, the extrapolated zero temperature result agrees with the result obtained from the expansion directly at $T = 0$. It should be noticed that when first-step RSB is included, the free energy is well behaved over a larger region closer to $T = 0$. This result may suggest that this divergent behaviour of the $1/M$ expansion may disappear for the presumably exact solution of infinite steps of RSB. If this is actually true, the coefficient of $1/\sqrt{M}$ in the

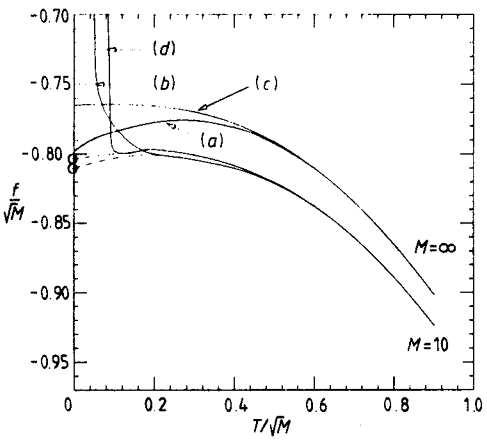

Figure 1. Rescaled free-energy density against the rescaled temperature in units of J_0 for $M = \infty$ (SK model) and $M = 10$ for no RS breaking ((*a*) and (*b*)) and first-step RSB ((*c*) and (*d*)). The broken line is the extrpolation to zero temperature. The values obtained directly at $T = 0$ is encircled.

expansion at $T=0$ should go to zero when the RSB step goes to infinity. Higher steps of RSB at zero temperature will be discussed in next section and will tend to support this scenario.

3. Zero temperature and higher-step RSB

For the zero temperature case, it is useful to define

$$\gamma_n(\{x_\alpha\}) \equiv g_n(\{\sigma_\alpha/\beta\})\,. \tag{14}$$

As shown in [12], $1/\sqrt{M}$ is the natural parameter to expand for $\beta \to \infty$. The details of the expansion and the RS and first-step RSB results were presented in [12]. For higher RSB, say pth-step RSB, the replica index α is parametrised [1, 14] as follows:

$$\alpha = (\{K\}, \gamma) \tag{15}$$

where

$$\gamma = 1, 2, \ldots, m_p \qquad \{K\} \equiv K^{(1)}, K^{(2)}, \ldots, K^{(p)}$$

with

$$\begin{aligned} K^{(1)} &= 1, 2, \ldots, n/m_1 \\ K^{(2)} &= 1, 2, \ldots, m_1/m_2 \\ &\vdots \\ K^{(p)} &= 1, 2, \ldots, m_{p-1}/m_p\,. \end{aligned} \tag{16}$$

The rule is that each new step introduces a new partitioning of the elements of the previous step. For example, for $q_{\alpha_1\alpha_2}$, at first-step, one has

$$(\alpha_1\alpha_2) = (11)(2) \tag{17}$$

At the second-step, the (11) cannot be further partitioned and remains the same while the (2) becomes

$$(2) \to \begin{pmatrix}2\\2\end{pmatrix}\begin{pmatrix}2\\11\end{pmatrix}. \tag{18}$$

The global order parameter $\gamma_n(\{x_\alpha\})$ depends on the replicated spins through the quantity $\sigma_{\{K\}} \equiv \sum_\gamma \sigma_{\{K\},\gamma}$. In what follows, the curly brackets in $\{K\}$ are dropped for convenience. As $\beta \to \infty$, $m_1, \ldots, m_p \to 0$ such that the βm are finite. We define

$$\mu_r \equiv \lim_{\beta\to\infty} \beta m_r \tag{19}$$

and denote $\mu_p \equiv \mu$. γ_n now satisfies the relation

$$\gamma_n(\{x_K\}) = \mathcal{N}^{-1} \int dJ\rho(J) \int \prod_K \frac{\mathrm{d}S_K}{2\pi} \gamma_n^M(\{\mathrm{i}S_K\}) \int \prod_K \mathrm{d}u_K \, \exp\Big(\mathrm{i}\sum_K S_K u_K \\ - x_K \operatorname{sgn}(J)\operatorname{sgn}(u_K)\min(|u_K|,|J|) + \mu\max(|u_K|,|J|)\Big) \tag{20}$$

where

$$\mathcal{N} = \int \prod_K \frac{dS_K}{2\pi} \gamma_n^M(\{iS_K\}) \int \prod_K du_K \exp\Big(i \sum S_K u_K + \mu |u_K| \Big) .$$

We introduce for convenience

$$b_K = \int du \exp\Big(iS_K u + \mu |u| \Big) \tag{21}$$

$$a_K = -\frac{1}{b_K} \int du \exp(iS_K u + \mu |u|) \operatorname{sgn}(u) \tag{22}$$

and

$$\langle A \rangle \equiv \frac{\int \prod_K (dS_K/2\pi) \gamma_n^M(\{iS_K\}) \prod_K b_K(S_K) A(\{S_K\})}{\int \prod_K (dS_K/2\pi) \gamma_n^M(\{iS_K\}) \prod_K b_K(S_K)} . \tag{23}$$

The $\langle \ldots \rangle$ is also expanded as

$$\langle A \rangle = \langle A \rangle_0 + \frac{\langle A \rangle_1}{\sqrt{M}} + \cdots . \tag{24}$$

Following similar steps as in [12], the free-energy density is expanded in powers of $1/\sqrt{M}$. For an arbitrary step of RSB, the result is

$$\frac{f}{\sqrt{M} J_0} = f_0 + \frac{f_1}{\sqrt{M}} + \frac{f_2}{M} + \mathcal{O}(\frac{1}{M\sqrt{M}}) \tag{25}$$

where

$$f_0 = \frac{\mu}{4} + \frac{\mu^2}{2n\beta} \sum_{(KK')} \langle a_K a_{K'} \rangle_0^2 - \frac{1}{n\beta} \ln \int \prod_K \frac{dS_K}{2\pi} b_K \times \exp\Big(-\sum_K S_K^2/2 - \sum_{(KK')} S_K S_{K'} \langle a_K a_{K'} \rangle_0 \Big) \tag{26}$$

$$f_1 = \frac{2}{3} \Big\langle \frac{1}{b_K} \Big\rangle_0^2 \tag{27}$$

and

$$f_2 = \frac{\mu^3}{24} - \frac{\mu}{4} - \Big(\frac{\mu^2}{12} + 1 \Big) \Big\langle \frac{1}{b_K} \Big\rangle_0 + \frac{\mu}{6} \Big\langle \frac{1}{b_K} \Big\rangle_0^2 + \frac{4\mu^2}{9} \Big\langle \frac{1}{b_K} \Big\rangle_0^3 - \frac{1}{6} \Big\langle \frac{S_K^2}{b_K} \Big\rangle_0 + \frac{4}{9} \Big\langle \frac{1}{b_K} \Big\rangle_0^2 \Big\langle \frac{S_K^2}{b_K} \Big\rangle_0 + \frac{\mu}{n\beta} \Big[\Big(\frac{5\mu^3}{6} - \frac{\mu}{2} \Big) \sum_{(KK')} \langle a_K a_{K'} \rangle_0^2 + \Big(\frac{\mu}{2} - \frac{\mu^3}{2} \Big) \sum_{(KK')} \langle a_K a_{K'} \rangle_1^2 + \frac{4\mu}{3} \sum_{(KK')} \langle a_K a_{K'} \rangle_0 \langle a_K b_{K'}^{-1} iS_{K'} \rangle_0$$

$$
\begin{aligned}
&-\frac{16\mu}{9}\left\langle\frac{1}{b_K}\right\rangle_0^2 \sum_{(KK')} \langle b_K^{-1} b_{K'}^{-1}\rangle_0 + \frac{8\mu}{3}\left\langle\frac{1}{b_K}\right\rangle_0 \sum_{(KK')} \langle a_K a_{K'}\rangle_1 \langle a_K b_{K'}^{-1} \mathrm{i} S_{K'}\rangle_0 \\
&+ 2\mu^2\left\langle\frac{1}{b_K}\right\rangle_0 \sum_{(KK')} \langle a_K a_{K'}\rangle_0^2 + 2\mu \sum_{(KK')} \langle a_K a_{K'}\rangle_0^2 \langle b_K^{-1} b_{K'}^{-1}\rangle_0 \\
&- 2\mu^2\left\langle\frac{1}{b_K}\right\rangle_0 \sum_{(KK')} \langle a_K a_{K'}\rangle_1^2 - 2\mu \sum_{(KK')} \langle a_K a_{K'}\rangle_1^2 \langle b_K^{-1} b_{K'}^{-1}\rangle_0 \\
&+ 4\mu^2\left\langle\frac{1}{b_K}\right\rangle_0 \sum_{(K_1K_2K_3)} \langle a_{K_1} a_{K_2}\rangle_1 \langle a_{K_1} a_{K_2} b_{K_3}^{-1}\rangle_0 \\
&- 3\mu^3 \sum_{(K_1K_2K_3)} \langle a_{K_1} a_{K_2}\rangle_0 \langle a_{K_2} a_{K_3}\rangle_1 \langle a_{K_1} a_{K_3}\rangle_1 \\
&+ 3\mu^3 \sum_{(K_1K_2K_3)} \langle a_{K_1} a_{K_2}\rangle_0 \langle a_{K_2} a_{K_3}\rangle_0 \langle a_{K_1} a_{K_3}\rangle_0 \\
&+ 6\mu^2 \sum_{(K_1K_2K_3)} \langle a_{K_1} a_{K_3}\rangle_0 \langle a_{K_2} a_{K_3}\rangle_0 \langle a_{K_1} a_{K_2} b_{K_3}^{-1}\rangle_0 \\
&- 6\mu^2 \sum_{(K_1K_2K_3)} \langle a_{K_1} a_{K_3}\rangle_1 \langle a_{K_2} a_{K_3}\rangle_1 \langle a_{K_1} a_{K_2} b_{K_3}^{-1}\rangle_0 \\
&- \frac{\mu^3}{2} \sum_{(K_1K_2K_3K_4)} \langle a_{K_1} a_{K_2} a_{K_3} a_{K_4}\rangle_0^2 \\
&+ 3\mu^3 \sum_{(K_1K_2K_3K_4)} \langle a_{K_1} a_{K_2}\rangle_0 \langle a_{K_3} a_{K_4}\rangle_0 \langle a_{K_1} a_{K_2} a_{K_3} a_{K_4}\rangle_0 \\
&- 3\mu^3 \sum_{(K_1K_2K_3K_4)} \langle a_{K_1} a_{K_2}\rangle_1 \langle a_{K_3} a_{K_4}\rangle_1 \langle a_{K_1} a_{K_2} a_{K_3} a_{K_4}\rangle_0 \Bigg] .
\end{aligned}
\tag{28}
$$

The limits $n \to 0$ and $\beta \to \infty$ are understood. Notice that the quantities that involve a single index K are independent of K. Also $\langle a_K a_{K'}\rangle$ and $\langle a_{K_1} a_{K_2} a_{K_3} a_{K_4}\rangle$ correspond to $q_{\alpha_1\alpha_2}$ and $q_{\alpha_1\alpha_2\alpha_3\alpha_4}$, respectively. $\langle a_K a_{K'}\rangle_1$ (i.e. $q^{(1)}_{\alpha_1\alpha_2}$) can be expressed in terms of the quantities of leading orders and thus one need only compute the $\langle\ldots\rangle_0$ averages. Finally the parameters μ should also be expanded

$$
\mu_r = \mu_r^{(0)} + \frac{\mu_r^{(1)}}{\sqrt{M}} + \cdots . \tag{29}
$$

The $\mu^{(0)}$ and $\langle a_K a_{K'}\rangle_0$ are evaluated from the extremisation condition of f_0. The fact that $\mu_r \neq \mu_r^{(0)}$ introduces a correction when f is evaluated with $\mu_r^{(0)}$. This correction can be calculated and expressed in terms of the derivatives of f_1 and f_0 evaluated at $\mu_r^{(0)}$ and contributes to the order $1/M$ term in the free energy.

For actual numerical evaluation, we considered second-step RSB. In this case, $q_{\binom{2}{2}} = 1$ at zero temperature, only $q_{\binom{2}{11}}$ and q_{11} are relevant and for simplicity they are denoted by q_2 and q_{11} respectively if no confusion arises. Similarly, for the q with four indices, only those with the indices $\binom{4}{1111}$, $\binom{22}{1111}$, $\binom{31}{1111}$, $\binom{211}{1111}$ and (1111) enter and again for simplicity they are denoted by (4), (22), (31), (211) and (1111), respectively. The free-energy density expansion is then obtained:

$$f_0 = \frac{\mu}{4}\left(1 - \eta q_{11}^{(0)2} - (1-\eta) q_2^{(0)2}\right) - \frac{1}{\eta\mu}\int Dz \ln \int Dy \left(\int Du\, e^{\mu|\theta|}\right)^{\eta} \tag{30}$$

where $\eta \equiv \mu_1/\mu_2$ and $\theta \equiv u\sqrt{1-q_2^{(0)}} + y\sqrt{q_2^{(0)} - q_{11}^{(0)}} + z\sqrt{q_{11}^{(0)}}$; the averages $\langle \ldots \rangle_0$ can be expressed in terms of nested integrals. For example

$$\left\langle \frac{1}{b_K} \right\rangle_0 = \frac{1}{\sqrt{2\pi(1-q_2^{(0)})}} \int Dz \frac{\int Dy\, e^{-\Phi/2} \left(\int Du\, e^{\mu|\theta|}\right)^{\eta-1}}{\int Dy \left(\int Du\, e^{\mu|\theta|}\right)^{\eta}} \tag{31}$$

where

$$\Phi \equiv \frac{1}{1-q_2^{(0)}}\left(y\sqrt{q_2^{(0)} - q_{11}^{(0)}} + z\sqrt{q_{11}^{(0)}}\right)^2 .$$

These averages at second-step RSB are listed in the appendix. f_1 is still given by (27). The expression of f_2 is too long and will be given in the appendix.

Firstly, $q_{11}^{(0)}$, $q_2^{(0)}$, $\mu_1^{(0)}$ and $\mu_2^{(0)}$ are given by the solution of the 4×4 nonlinear equations which are obtained from extremising f_0. Then $q_{\alpha\beta\gamma\delta}^{(0)}$ and other averages are evaluated using the integral expressions in the appendix. $q_{\alpha\beta}^{(1)}$ is then evaluated. The correction due to the fact that $\mu_r \neq \mu_r^{(0)}$ is taken care of. Finally, the numerical values of f_0, f_1 and f_2 can then be computed. By setting $q_2 = q_{11}$, the first-step RSB expressions are recovered and we also confirm the numerical values given in [12]. Our result with second-step RSB is

$$\frac{f}{\sqrt{M} J_0} = -0.7636 + \frac{0.0026}{\sqrt{M}} - \frac{0.434}{M} + \mathcal{O}\left(\frac{1}{M\sqrt{M}}\right). \tag{32}$$

Table 1 summarises the results of RS and RSB at first and second steps. Our leading term in the free energy agrees with Parisi's result [1] for the SK model. The coefficient of $1/\sqrt{M}$ (i.e. f_1) decreases as a higher step of RSB is used and indeed is quite close to zero in our second-step result. We speculate that as the step of RSB goes to infinity, $f_1 \to 0$. The divergence as $T \to 0$ in the $1/M$ expansion at finite temperature may be an artifact of the finite step RSB. However, we are still lacking a rigorous proof.

Table 1. Coefficients of the free-energy expansion at zero temperature.

Solution	f_0	f_1	f_2
RS	−0.798	0.106	−0.437
First RSB	−0.765	0.010	−0.390
Second RSB	−0.7636	0.0026	−0.434

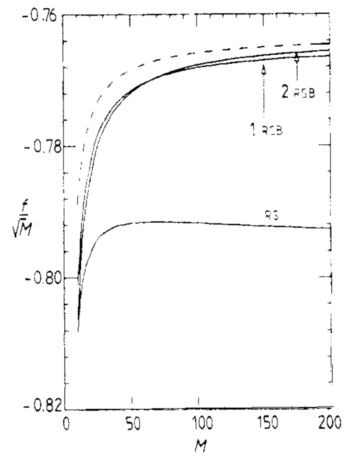

Figure 2. Rescaled ground state energy against M. Dashed line is the empirical results from equation (33). Full curves are $1/\sqrt{M}$ expansion results with no, first-step and second-step RSB.

Finally our results can be compared with the numerical simulation [7] results of the ground state energy. Figure 2 summarises the various results. The simulation results, denoted by the broken curve, fits the empirical formula

$$\frac{E_0}{J_0\sqrt{M}} = -\frac{M+1}{2\sqrt{M}}\frac{c}{\sqrt{M-1+c^2}} \tag{33}$$

with $c = 1.5266$ in the range $2 < M < 20$. Our second-step RSB results up to order $1/M$ start to show improvement over the first-step results (i.e. closer to the empirical values) for $M \geq 60$. For moderate values of M ($10 \leq M \leq 60$), since the second leading coefficient (f_1) is very close to zero, higher orders needed to be included. One way to include some of the higher-order terms is instead of calculating the free energy by $f(\mu_r = \mu_r^{(0)})$ + corrections $\mathcal{O}(1/M)$, to compute $f(\mu_r = \mu_r^{(0)} + \mu_r^{(1)}/\sqrt{M})$. By doing so, our second-step RSB results in the $10 \leq M \leq 60$ range are improved and are slightly closer to the empircal results than the first-step results. (A similar procedure performed on the first-step results show negligible changes to the free energy.) However, one should bear in mind that the difference between the first-step and second-step results are about 0.5% for moderate values of M, which is of the same order as the uncertainties in fitting the simulation results to the empirical formula.

4. Conclusion

The large connectivity expansion method with RSB is used to obtain the free energies of the Ising spin-glass on lattices with fixed and finite valence. At finite temperatures, the free energy as a function of temperature is calculated at first-step RSB by $1/M$ expansion. As in the RS case, the free energy appears to diverge when $T \to 0$, but is well behaved over a larger range closer to $T = 0$. Higher RSB are discussed for the $1/\sqrt{M}$ expansion at $T = 0$. Expression for the free-energy expansion is derived for an arbitrary step of RSB. Numerical results are obtained at second-step RSB. The coefficient of $1/\sqrt{M}$ is close to zero suggesting that it might go to zero at infinite-step of RSB and thus the apparent divergence in the $1/M$ expansion at finite temperature might disappear in the exact infinite-step RSB solution. The remaining deviation from numerical simulations of graph partitioning are probably due in part to higher order $1/\sqrt{M}$ corrections and in part to inaccuracies in the simulations themselves which tend to overestimate the cost of the partition.

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Appendix

The expression of f_2 at the second step of RSB is

$$\begin{aligned}
f_2 = {} & \frac{\mu^3}{24} - \frac{\mu}{4} - \left(\frac{\mu^2}{12}+1\right)\left\langle\frac{1}{b_K}\right\rangle_0 + \frac{\mu}{6}\left\langle\frac{1}{b_K}\right\rangle_0^2 + \frac{4\mu^2}{9}\left\langle\frac{1}{b_K}\right\rangle_0^3 \\
& - \frac{1}{6}\left\langle\frac{S_K^2}{b_K}\right\rangle_0 + \frac{4}{9}\left\langle\frac{1}{b_K}\right\rangle_0^2\left\langle\frac{S_K^2}{b_K}\right\rangle_0 \\
& + \left(\frac{5\mu^3}{6} - \frac{\mu}{2}\right) DB(q_{11}^{(0)2}, q_2^{(0)2}) + \left(\frac{\mu}{2} - \frac{\mu^3}{2}\right) DB(q_{11}^{(1)2}, q_2^{(1)2}) \\
& + \frac{4\mu}{3} DB(q_{11}^{(0)}S_{11}, q_2^{(0)}S_2) - \frac{16\mu}{9}\left\langle\frac{1}{b_K}\right\rangle_0^2 DB(B_{11}, B_2) \\
& + \frac{8\mu}{3}\left\langle\frac{1}{b_K}\right\rangle_0 DB(q_{11}^{(1)}S_{11}, q_2^{(1)}S_2) + 2\mu^2\left\langle\frac{1}{b_K}\right\rangle_0 DB(q_{11}^{(0)2}, q_2^{(0)2}) \\
& + 2\mu DB(q_{11}^{(0)2}B_{11}, q_2^{(0)2}B_2) - 2\mu^2\left\langle\frac{1}{b_K}\right\rangle_0 DB(q_{11}^{(1)2}, q_2^{(1)2}) \\
& - 2\mu DB(q_{11}^{(1)2}B_{11}, q_2^{(1)2}B_2) \\
& + 4\mu^2\left\langle\frac{1}{b_K}\right\rangle_0 TR(q_{11}^{(1)}A_{111}, q_2^{(1)}A_3, q_2^{(1)}A_{21}, q_{11}^{(1)}A_{12}) \\
& - 3\mu^3 TR(q_{11}^{(0)}q_{11}^{(1)2}, q_2^{(0)}q_2^{(1)2}, q_2^{(0)}q_{11}^{(1)2}, q_{11}^{(0)}q_2^{(1)}q_{11}^{(1)}) \\
& + 3\mu^3 TR(q_{11}^{(0)3}, q_2^{(0)3}, q_2^{(0)}q_{11}^{(0)2}, q_{11}^{(0)2}q_2^{(0)}) \\
& + 6\mu^2 TR(q_{11}^{(0)2}A_{111}, q_2^{(0)2}A_3, q_{11}^{(0)2}A_{21}, q_{11}^{(0)}q_2^{(0)}A_{12}) \\
& - 6\mu^2 TR(q_{11}^{(1)2}A_{111}, q_2^{(1)2}A_3, q_{11}^{(1)2}A_{21}, q_{11}^{(1)}q_2^{(1)}A_{12}) \\
& - \frac{\mu^3}{2} QD(q_{1111}^2, q_4^2, q_{22}^2, q_{22}^2, q_{112}^2, q_{112}^2, q_{13}^2) \\
& + 3\mu^3 QD(q_{11}^{(0)2}q_{1111}, q_2^{(0)2}q_4, q_2^{(0)2}q_{22}, q_{11}^{(0)2}q_{22}, q_{11}^{(0)}q_2^{(0)}q_{112}, q_{11}^{(0)2}q_{112}, q_{11}^{(0)}q_2^{(0)}q_{13}) \\
& - 3\mu^3 QD(q_{11}^{(1)2}q_{1111}, q_2^{(1)2}q_4, q_2^{(1)2}q_{22}, q_{11}^{(1)2}q_{22}, q_{11}^{(1)}q_2^{(1)}q_{112}, q_{11}^{(1)2}q_{112}, q_{11}^{(1)}q_2^{(1)}q_{13})
\end{aligned} \tag{A1}$$

where

$$DB(x,y) \equiv \frac{1}{2}(-\eta x + (\eta-1)y)$$

$$TR(w,x,y,z) \equiv \frac{\eta^2 w}{3} + \frac{(\eta-1)(\eta-2)x}{6} - \frac{\eta(\eta-1)y}{6} - \frac{\eta(\eta-1)z}{3}$$

$$QD(t,u,v,w,x,y,z) \equiv -\frac{\eta^3 t}{4} + \frac{(\eta-1)(\eta-2)(\eta-3)u}{24} - \frac{\eta(\eta-1)^2 v}{24} - \frac{\eta(\eta-1)^2 w}{12} + \frac{\eta^2(\eta-1)x}{6} + \frac{\eta^2(\eta-1)y}{3} - \frac{\eta(\eta-1)(\eta-2)z}{6}.$$

The expressions for the leading order averages $\langle\ldots\rangle_0$ at second RSB step are given below. $\langle b_K^{-1} b_{K'}^{-1}\rangle_0$, $\langle a_K b_{K'}^{-1} \mathrm{i} S_{K'}\rangle_0$ and $\langle a_{K_1} a_{K_2} b_{K_3}^{-1}\rangle_0$ are denoted by B, S and A, respectively, and $\nu(y,z) \equiv y\sqrt{q_2^{(0)} - q_{11}^{(0)}} + z\sqrt{q_{11}^{(0)}}$, $c_1 \equiv 1/\sqrt{2\pi(1-q_2^{(0)})}$ and $c_2 \equiv 1/\sqrt{2\pi(1-q_2^{(0)})^3}$.

$$\left\langle \frac{1}{b_K} \right\rangle_0 = c_1 \int Dz \, \frac{\int Dy \, \mathrm{e}^{-\Phi/2} \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta-1}}{\int Dy \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta}}$$

$$\left\langle \frac{S_K^2}{b_K} \right\rangle_0 = c_2 \int Dz \, \frac{\int Dy \, (1-\Phi) \mathrm{e}^{-\Phi/2} \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta-1}}{\int Dy \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta}}$$

$$B_{11} = c_1^2 \int Dz \left(\frac{\int Dy \, \mathrm{e}^{-\Phi/2} \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta-1}}{\int Dy \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta}} \right)^2$$

$$B_2 = c_1^2 \int Dz \, \frac{\int Dy \, \mathrm{e}^{-\Phi} \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta-2}}{\int Dy \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta}}$$

$$S_{11} = -c_2 \int Dz \left[\frac{\left[\int Dy \, \nu(y,z) \mathrm{e}^{-\Phi/2} \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta-1}\right]}{\left[\int Dy \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta}\right]^2} \times \int Dy \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta-1} \int Du \, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|} \right]$$

$$S_2 = -c_2 \int Dz \, \frac{\int Dy \, \nu(y,z) \mathrm{e}^{-\Phi/2} \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta-2} \left(\int Du \, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|}\right)}{\int Dy \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta}}$$

$$A_{111} = c_1 \int Dz \left\{ \frac{\left[\int Dy \, \mathrm{e}^{-\Phi/2} \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta-1}\right]}{\left[\int Dy \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta}\right]^3} \times \left[\int Dy \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta-1} \int Du \, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|} \right]^2 \right\}$$

$$A_{21} = c_1 \int Dz \left\{ \frac{\left[\int Dy \, \mathrm{e}^{-\Phi/2} \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta-1}\right]}{\left[\int Dy \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta}\right]^2} \times \left[\int Dy \left(\int Du \, \mathrm{e}^{\mu|\theta|}\right)^{\eta-2} \left(\int Du \, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|}\right)^2 \right] \right\}$$

$$A_{12} = c_1 \int Dz \left[\frac{\left[\int Dy\, \mathrm{e}^{-\Phi/2} \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta-2} \left(\int Du\, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|}\right)\right]}{\left[\int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta}\right]^2} \right.$$

$$\left. \times \int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|} \right)^{\eta-1} \left(\int Du\, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|} \right) \right]$$

$$A_3 = c_1 \int Dz\, \frac{\int Dy\, \mathrm{e}^{-\Phi/2} \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta-3} \left(\int Du\, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|}\right)^2}{\int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta}}$$

$$q_{1111} = \int Dz \left(\frac{\int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta-1} \left(\int Du\, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|}\right)}{\int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta}} \right)^4$$

$$q_{112} = \int Dz \left[\frac{\left[\int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta-1} \left(\int Du\, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|}\right)\right]^2}{\left[\int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta}\right]^3} \right.$$

$$\left. \times \int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|} \right)^{\eta-2} \left(\int Du\, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|} \right)^2 \right]$$

$$q_{13} = \int Dz\, \frac{\left[\int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta-1} \left(\int Du\, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|}\right)\right]}{\left[\int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta}\right]^2}$$

$$\times \int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|} \right)^{\eta-3} \left(\int Du\, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|} \right)^3 \Bigg]$$

$$q_{22} = \int Dz \left(\frac{\int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta-2} \left(\int Du\, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|}\right)^2}{\int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta}} \right)^2$$

$$q_4 = \int Dz\, \frac{\int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta-4} \left(\int Du\, \mathrm{sgn}\,(\theta) \mathrm{e}^{\mu|\theta|}\right)^4}{\int Dy \left(\int Du\, \mathrm{e}^{\mu|\theta|}\right)^{\eta}} \,.$$

Note that A denotes $\langle a_{K_1} a_{K_2} b_{K_3}^{-1} \rangle_0$, A_{21} stands for the case where K_1 and K_2 are in the same partition (A_{12} otherwise) and $A_{21} \neq A_{12}$.

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